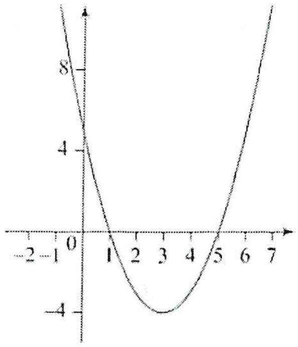
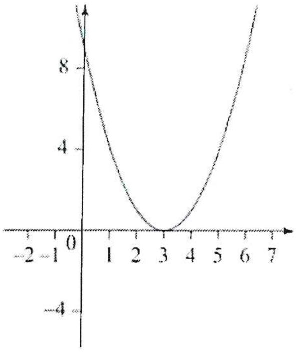
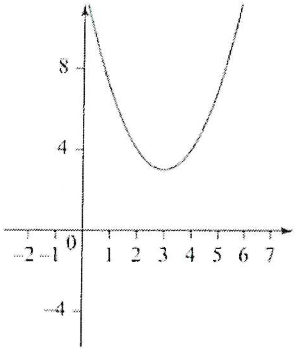
Quadratics - Discriminant Analysis



If $\Delta > 0$, the quadratic $y = ax^2 + bx + c$ has two distinct roots and the curve crosses the x-axis at two distinct points.



If $\Delta = 0$, the quadratic $y = ax^2 + bx + c$ has one repeated root and the x-axis is a tangent to the curve at this point.



If $\Delta < 0$, the quadratic $y = ax^2 + bx + c$ has no (real) roots and the curve does not cross the x-axis at any point.

1.

The quadratic equation $(k+1)x^2 + 12x + (k-4) = 0$ has real roots.

(a) Show that $k^2 - 3k - 40 \leqslant 0$. *(3 marks)*

(b) Hence find the possible values of k. *(4 marks)*

2.

(a) (i) Express $2x^2 - 20x + 53$ in the form $2(x-p)^2 + q$, where p and q are integers. *(2 marks)*

(ii) Use your result from part **(a)(i)** to explain why the equation $2x^2 - 20x + 53 = 0$ has no real roots. *(2 marks)*

(b) The quadratic equation $(2k-1)x^2 + (k+1)x + k = 0$ has real roots.

(i) Show that $7k^2 - 6k - 1 \leqslant 0$. *(4 marks)*

(ii) Hence find the possible values of k. *(4 marks)*

3.

The quadratic equation $x^2 + (m+4)x + (4m+1) = 0$, where m is a constant, has equal roots.

(a) Show that $m^2 - 8m + 12 = 0$. *(3 marks)*

(b) Hence find the possible values of m. *(2 marks)*

4.

Show that the quadratic equation

$$(k+1)x^2+2kx+k=1$$

has two distinct real roots for all values of k, except for one value which must be stated.

5.

Find the range of values that the constant k can take so that

$$2x^2+(k+2)x-k=0$$

has two distinct real roots.

6.

Find the possible solutions of the quadratic equation

$$x^2+(3-m)x+5=m^2,$$

where m is a constant, given that it has repeated roots.

7.

$$x^2-4ax+2b+1=0.$$

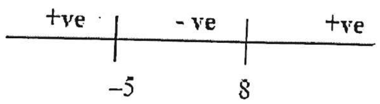
The above quadratic equation, where a and b are constants, has no real solutions.

Show clearly that

$$b>\tfrac{1}{2}(2a+1)(2a-1).$$

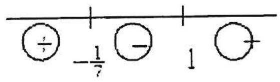
Quadratics - Discriminant Analysis Solutions

1.

(a)	$b^2 - 4ac \;=\; 144 - 4(k+1)(k-4)$	M1		Clear attempt at $b^2 - 4ac$ Condone slip in one term of expression
	Real roots when $b^2 - 4ac \geqslant 0$	B1		Not just a statement, must involve k
	$36 - (k^2 - 3k - 4) \geqslant 0$ $\Rightarrow k^2 - 3k - 40 \leqslant 0$	A1	3	AG (watch signs carefully)
(b)	$(k-8)(k+5)$	M1		Factors attempt or formula
	Critical points 8 and -5	A1		
	Sketch or sign diagram correct, must have 8 and -5	M1		
	$-5 \leqslant k \leqslant 8$	A1	4	
	A0 for $-5 < k < 8$ or two separate inequalities unless word AND used			

2.

(a)(i)	$2(x-5)^2$	B1		$p = 5$
	$+3$	B1	2	$q = 3$
(ii)	Stating both $(x-5)^2 \geqslant 0$ and $3 > 0$ $\Rightarrow 2x^2 - 20x + 53 > 0$ or $2(x-5)^2 + 3 > 0$	M1		FT their p & q, but must have $q > 0$
	$\Rightarrow 2x^2 - 20x + 53 = 0$ has no real roots	A1cso	2	Must have statement and correct p & q.
(b)(i)	$b^2 - 4ac = (k+1)^2 - 4k(2k-1)$	M1		Condone one slip (including x is one slip)
	$= -7k^2 + 6k + 1$	A1		Condone recovery from missing brackets
	real roots $\Rightarrow b^2 - 4ac \geqslant 0$ $-7k^2 + 6k + 1 \geqslant 0$	B1√		Their discriminant $\geqslant 0$ (in terms of k) Need not be simplified & may earn earlier
	$\Rightarrow 7k^2 - 6k - 1 \leqslant 0$	A1cso	4	AG (must see sign change)
(ii)	$(7k+1)(k-1)$	M1		Correct factors or correct use of formula
	Critical values $k = 1, -\dfrac{1}{7}$	A1		May score M1, A1 for correct critical values seen as part of incorrect final answer with or without working.

Use of sign diagram or sketch  	M1		If previous A1 earned, sign diagram or sketch must be correct for M1 Otherwise M1 may be earned for an attempt at the sketch or sign diagram using *their* critical values.
$-\frac{1}{7}\leqslant k\leqslant 1$ Full marks for correct answer NMS Condone $-\frac{2}{14}$ throughout Condone $k\geqslant-\frac{1}{7}$ AND $k\leqslant 1$ for full marks *Take their final line as their answer.*	A1	4	$\left(-\frac{1}{7}<k<1\right)$, $\left(k\geqslant-\frac{1}{7}\text{ OR } k\leqslant 1\right)$, $\left(k\geqslant-\frac{1}{7},\ k\leqslant 1\right)$ score M1A1M1A0 *Answer only of* $k<-\frac{1}{7}$, $k<1$ etc scores M1, A1, M0 since the critical values are evident. *Answer only of* $\frac{1}{7}\leqslant k\leqslant 1$ etc scores M0, M0 since the critical values are not both correct.

3.

(a)	$(m+4)^2=m^2+8m+16$	B1		Condone $4m+4m$
	$b^2-4ac=(m+4)^2-4(4m+1)=0$	M1		b^2-4ac (attempted and involving m's and no x's) or $b^2-4ac=0$ stated
	$m^2+8m+16-16m-4=0$ $\Rightarrow m^2-8m+12=0$	A1	3	AG (be convinced – all working correct- $=0$ appearing more than right at the end)
(b)	$(m-2)(m-6)=0$	M1		Attempt at factors or quadratic formula
	$m=2$, $m=6$	A1	2	SC B1 for 2 or 6 only without working
	Total		5	

4.

$(k+1)x^2+2kx+k=1$
$(k+1)x^2+2kx+(k-1)=0$

$b^2-4ac=(2k)^2-4(k+1)(k-1)$
$=4k^2-4(k^2-1)$
$=4k^2-4k^2+4$
$=4$
>0

$\therefore$ ALWAYS TWO DISTINCT ROOTS UNLESS $k=-1$, BECAUSE QUADRATIC IS LINEAR!

5.

$2x^2+(k+2)x+k=0$

Two distinct roots $\Rightarrow b^2-4ac>0$

$\Rightarrow (k+2)^2-4\times 2\times k>0$

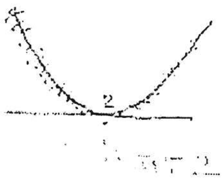
$\Rightarrow (k+2)^2-8k>0$

$\Rightarrow k^2+4k+4-8k>0$

$\Rightarrow k^2-4k+4>0$

$\Rightarrow (k-2)^2>0$

C.V $=2$



$\therefore x\in\mathbb{R}, x\neq 2$

6.

$x^2+(3-m)x+5=m^2$

$x^2+(3-m)x+(5-m^2)=0$

Repeated roots $\Rightarrow b^2-4ac=0$

$\Rightarrow (3-m)^2-4\times 1\times(5-m^2)=0$

$\Rightarrow 9-6m+m^2-20+4m^2=0$

$\Rightarrow 5m^2-6m-11=0$

$\Rightarrow (5m-11)(m+1)=0$

$\Rightarrow m=-1$ or $m=\frac{11}{5}$

- If $m=-1$

$x^2+(3-m)x+(5-m^2)=0$

$x^2+4x+4=0$

$(x+2)^2=0$

$x=-2$

- If $m=\frac{11}{5}$

$x^2+(3-m)x+(5-m^2)=0$

$x^2+(3-\frac{11}{5})x+(5-\frac{121}{25})=0$

$x^2+\frac{4}{5}x+\frac{4}{25}=0$

$25x^2+20x+4=0$

$(5x+2)^2=0$

$x=-\frac{2}{5}$

7.

$x^2 - 4ax + 2b+1 = 0$

(CHOOSE CAPITALS FOR DISCRIMINANT TO AVOID CONFUSION

NO REAL SOLUTIONS $\Rightarrow B^2 - 4AC < 0$

$$\Rightarrow (-4a)^2 - 4 \times 1 \times (2b+1) < 0$$

$$\Rightarrow 16a^2 - 4(2b+1) < 0$$

$$\Rightarrow 4a^2 - (2b+1) < 0$$

$$\Rightarrow -(2b+1) < -4a^2$$

$$\Rightarrow 2b+1 > 4a^2$$

$$\Rightarrow 2b > 4a^2 - 1$$

$$\Rightarrow 2b > (2a-1)(2a+1)$$

$$\Rightarrow b > \frac{1}{2}(2a-1)(2a+1)$$